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# Construction of Compactly Supported *M*-Band Wavelets

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In this paper, we consider the asymptotic regularity of Daubechies scaling functions and construct examples of $M$-band scaling functions which are both orthonormal and cardinal for $M \geq 3$. © 1999 Academic Press

*Key Words:* multiresolution; $M$-band scaling function; $M$-band wavelets; cardinal function; orthonormality.

## 1. INTRODUCTION

Let $M \geq 2$ be a fixed positive integer. A family of closed subspaces $V_j$, $j \in Z$, of $L^2$, the space of all square integrable functions on the real line, is said to be a *multiresolution* of $L^2$ if the following conditions hold:

(i) $V_j \subset V_{j+1}$, and $f \in V_j$ if and only if $f(M\cdot) \in V_{j+1}$ for all $j \in Z$;

(ii) $\bigcup_{j \in Z} V_j$ is dense in $L^2$ and $\bigcap_{j \in Z} V_j = \emptyset$;

(iii) there exists a function $\phi$ in $V_0$ such that $\{\phi(\cdot - k);\ k \in Z\}$ is a Riesz basis of $V_0$.

Here we say that $\{\phi(\cdot - k);\ k \in Z\}$ is a *Riesz basis* of $V_0$ if there exist constants $0 < A \leq B < \infty$ such that

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$$A\Big(\sum_{k\in Z}|d(k)|^2\Big)^{1/2} \le \left(\int_R \Big|\sum_{k\in Z} d(k)\phi(x-k)\Big|^2 dx\right)^{1/2} \le B\Big(\sum_{k\in Z}|d(k)|^2\Big)^{1/2}$$

for all square summable sequences $\{d(k)\}$ and $V_0$ is spanned by $\{\phi(\cdot - k); k \in Z\}$. When $A = B = 1$, we say that $\{\phi(\cdot - k); k \in Z\}$ is an *orthonormal basis* of $V_0$, or simply say that $\phi$ is *orthonormal.*

The concept of multiresolution was introduced by Meyer and Mallat (see [4, 7, 22]). The multiresolution above generalizes the one introduced by Meyer and Mallat, but simplifies the one by Devore *et al.* [3] for our purpose.

From the definition of a multiresolution, the function $\phi$ in (iii), which is called *M-band scaling function* or *scaling function* for short, satisfies the refinement equation

$$\phi(x) = \sum_{k\in Z} c_k\phi(Mx-k), \tag{1}$$

where $\{c_k\}$, which is called the *mask* of the above refinement equation, satisfies $\sum_{k\in Z} c_k = M$. In this paper, we always assume that $\phi$ is compactly supported and that the mask of the refinement equation (1) has finite length.

Let

$$H(z) = \frac{1}{M}\sum_{k\in Z} c_k z^k \tag{2}$$

be the *symbol* of the refinement equation (1), or of the refinable function $\phi$. Then $H(z)$ is a Laurent polynomial. By taking Fourier transform at both sides of (1), we have

$$\hat{\phi}(M\xi) = H(e^{-i\xi})\hat{\phi}(\xi).$$

Here the Fourier transform $\hat{f}$ of an integrable function is defined by

$$\hat{f}(\xi) = \int_R e^{-ix\xi} f(x)\,dx,$$

and the Fourier transform for a compactly supported distribution is understood as usual.

There is considerable literature on the problem under which condition on $\phi$ in (1) or under which condition on the mask $\{c_k\}$ in (1) the family of spaces $V_j$ spanned by $\{M^{j/2}\phi(M^j\cdot-k); k\in Z\}$ is a multiresolution of $L^2$ (see, for instance, [3] and references therein). In this paper, we give the following restriction on $\phi$: There exists a compactly supported distribution $\eta$, which is the solution of another refinement equation

$$\begin{aligned} \eta(x) &= \sum_{k\in Z} b_k\eta(Mx-k) \\ \hat{\eta}(0) &= 1, \end{aligned} \tag{3}$$

where the mask $\{b_k\}$ has finite length and $\sum_{k\in Z} b_k = M$, such that $\eta(\phi(\cdot - t))$ is well defined for all $t \in R$ and

$$f(x) = \sum_{k\in Z} \eta(f(\cdot + k))\phi(x-k) \tag{4}$$

for every function $f$ in the space spanned by $\{\phi(\cdot - k)\}$. Obviously the distribution $\eta$ such that (3) and (4) hold is not unique in general. The above restriction on $\phi$ does not restrict us very much. Examples of $\eta$ such that (3) and (4) hold are $\phi$ when $\{\phi(\cdot - k)\}$ is an orthonormal basis of $V_0$ [7]; $\phi$ when $\phi$ and $\phi$ are biorthogonal [6]; $\delta$, the delta distribution, when $\phi$ is *cardinal* which means that $\phi$ is continuous at integer points and $\phi(k) = 0$ for $k \in Z$ except $\phi(0) = 1$ [21]; and $\phi^*$ when $\phi^*$ is the dual of $\phi$ defined by

$$\hat{\phi}^*(\xi) = \frac{\hat{\phi}(\xi)}{\sum_{k\in Z} |\hat{\phi}(\xi + 2k\pi)|^2}.$$

In this paper, we only consider the two useful cases: $\eta = \phi$ and $\eta = \delta$.

A central problem in wavelet analysis is to construct appropriate wavelet bases. On the other hand, almost all interesting and useful wavelet bases can be constructed from multiresolutions, and the scaling functions play an essential role in multiresolution. So there is considerable literature devoted to the construction of various kind of scaling functions and the properties of such scaling functions. For $M=2$, the popular wavelets are Daubechies' orthonormal wavelets; biorthogonal wavelets by Cohen, Daubechies and Feauveau; spline wavelets by Chui and Wang; etc. For $M \geq 3$, multiresolution and $M$-band wavelets are well studied (see [1, 10–15, 17–19, 24, 25, 29], etc.). Let

$$a_{M,N}(s) = \sum_{s_1+\cdots+s_{M-1}=s} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j} \left(2 \sin \frac{j\pi}{M}\right)^{-2s_j} \tag{5}$$

and

$$P_{M,N}(z) = \sum_{s=0}^{N-1} a_{M,N}(s)(2 - z - z^{-1})^s. \tag{6}$$

Then the following explicit formula to the symbol of orthonormal scaling function is proved by Heller [12] and independently by the authors [2].

THEOREM 1. *Let $H(z) = ((1 - z^M)/M(1-z))^N Q(z)$ be the symbol of an orthonormal scaling function with compact support and $P_{M,N}(z)$ be defined by* (6). *Then*

$$Q(z)Q(z^{-1}) = P_{M,N}(z) + (1-z)^N(1-z^{-1})^N \sum_{s=1}^{M-1} z^s R_s(z^M), \tag{7}$$

*where* $R_s(z^{-1}) = zR_{M-s}(z)$ *and*

$$P_{M,N}(z) + (1 - z)^N(1 - z^{-1})^N \sum_{s=1}^{M-1} z^s R_s(z^M) \geq 0$$

*for all* $z = e^{i\xi}$.

It is known that the solution of the refinement equation (1) with symbol $H(z)$ satisfying (7) is not always an orthonormal scaling function of a multiresolution. But when $R_s = 0$ in (7) for $1 \leq s \leq M - 1$, it becomes true. In particular, the corresponding solution, which we denote $\phi_{M,N}$, is an orthonormal scaling function and is the one constructed by Daubechies in [8] when $M = 2$. So we call $\phi_{M,N}$ the Daubechies scaling function even when $M \geq 2$.

The first conclusion in this paper is on the asymptotic regularity of $\phi_{M,N}$. The regularity of refinable function is an important factor in application. The regularity widely considered are the Sobolev exponent, Hölder exponent and Besov exponent (see [5, 7, 9, 16, 20, 22, 26–28], etc.) The Sobolev exponent $s_p(f)$, $0 < p < \infty$, is defined by

$$s_p(f) = \sup\left\{ s;\ \int_R |\hat{f}(\xi)|^p(1 + |\xi|^p)^s d\xi < \infty \right\}$$

for $0 < p < \infty$ and

$$s_\infty(f) = \sup\{s;\ \hat{f}(\xi)(1 + |\xi|)^s \text{ is bounded}\}.$$

The Hölder exponent $\alpha(f)$ is defined by

$$\alpha(f) = \sup\{\alpha;\ f \in C^\alpha\},$$

where $C^\alpha$ is the usual Hölder class. For compactly supported function $f$, the Sobolev exponent and Hölder exponent are closed related,

$$s_p(f) - \frac{1}{p} \leq s_q(f) - \frac{1}{q},$$

for $0 < p \leq q \leq \infty$ and

$$s_1(f) \leq \alpha(f) \leq s_\infty(f).$$

In this paper, we use the Sobolev exponent $s_\infty(f)$ as the index to the regularity of a function $f$. The Sobolev exponent $s_\infty(f)$ was introduced by Cohen and Conze in [5] and $s_p(f)$, $1 < p < \infty$, was introduced by Herve in [16]. For simplicity, we write $s_\infty(\phi_{M,N})$ as $\alpha_{M,N}$.

When $M = 2$, Volker proved that

$$N - \frac{\ln|P_{2,N}(e^{2\pi i/3})|}{\ln 4} - \frac{1}{2} \le s_2(\phi_{2,N}) \le N - \frac{\ln|P_{2,N}(e^{2\pi i/3})|}{\ln 4}$$

in [28] and

$$\alpha(\phi_{2,N}) = \left(1 - \frac{\ln 3}{\ln 4}\right)N + o(N)$$

in [27]. The second result was also proved by Cohen and Conze in [5]. For $M = 3, 4$, Heller and Wells proved similar estimates to $s_2(\phi_{M,N})$ [15]. In this paper, we will prove

THEOREM 2. *Let $\phi_{M,N}$ and $\alpha_{M,N}$ be defined as above. Then there exists a constant $C$ independent of $N$ such that*

$$\left|\alpha_{M,N} - \frac{\ln N}{4 \ln M}\right| \le C \tag{8}$$

*when $M$ is odd and*

$$\left|\alpha_{M,N} - \frac{-4N \ln \sin \dfrac{M\pi}{2M+2} + \ln N}{4 \ln M}\right| \le C \tag{9}$$

*when $M$ is even.*

From our result, we see that the Sobolev exponent of scaling functions $\phi_{M,N}$ increases almost linearly to $N$ when $M$ is even, and increases only at the rate of $\ln N$ when $M$ is odd. Such a phenomenon was also observed by Heller and Wells [15] and Villemoes according to [7, p. 320]. In 1995, Shi and the third author constructed a class of $M$-band scaling functions which Sobolev exponent grows proportionally to their symbol support widths for all $M \ge 3$ [23]. This also gives an affirmative answer to the remark in [7, p. 338].

The second part of this paper is devoted to the problem $\eta = \delta$ in (3) and (4). In this case we see that $\phi$ is continuous at all integer points and $\phi(k)$ takes value 1 when $k = 0$ and 0 when $k \ne 0$. We call such a function as *cardinal function.*

In 1994, Lewis noticed the importance of constructing a cardinal scaling function. An easy way to construct a cardinal function is by self-convolution of an orthonormal function. Specifically, the function $\Phi$ defined by

$$\Phi(x) = \int_R \phi(y - x)\phi(y)\,dy$$

is a cardinal function when $\phi$ is an orthonormal function. The construction of a cardinal scaling function for $M \ge 3$ was considered by Heller [11] and the authors [2] independently. The following formula is taken from [2].

Define

$$Q_{M,N}(z) = z^{-(M-1)N/2} \sum_{s=0}^{[(N-1)/2]} b_{M,N}(s)(2 - z - z^{-1})^s \tag{10}$$

when $M$ is odd,

$$Q_{M,N}(z) = z^{-(M-1)N/2} \sum_{0 \le s \le N/2-1} b_{M,N}(s)(2 - z - z^{-1})^s \tag{11}$$

when $M$ and $N$ are even, and

$$Q_{M,N}(z) = z^{-(M-1)N/2-1/2} \Bigg( \sum_{0 \le s \le (N-1)/2} b_{M,N}(s)(2 - z - z^{-1})^s - \frac{1-z}{2} \sum_{0 \le s \le (N-3)/2} b_{M,N}(s)(2 - z - z^{-1})^s \Bigg) \tag{12}$$

when $M$ is even and $N$ is odd, where

$$b_{M,N}(s) = \sum_{s_1 + \cdots + s_{(M-1)/2} = s} \prod_{j=1}^{(M-1)/2} \binom{N + s_j - 1}{s_j} \left( 2 \sin \frac{j\pi}{M} \right)^{-2s_j}$$

when $M$ is odd and

$$b_{M,N}(s) = \sum_{s_1 + \cdots + s_{M/2} = s} \prod_{j=1}^{M/2-1} \binom{N + s_j - 1}{s_j} \left( 2 \sin \frac{j\pi}{M} \right)^{-2s_j} \times \binom{[(N+1)/2] + s_{M/2} - 1}{s_{M/2}} 4^{-s_{M/2}}. \tag{13}$$

when $M$ is even. Then we have

THEOREM 3. *Let $Q_{M,N}(z)$ be defined as in* (10)–(12). *If a compactly supported continuous solution $\phi$ of* (1) *is cardinal, then its symbol $H(z) = ((1 - z^M)/(M - Mz))^N Q(z)$ satisfies*

$$Q(z) = Q_{M,N}(z) + (1 - z)^N \sum_{s=1}^{M-1} z^s R_s(z^M) \tag{14}$$

*for some Laurent polynomials $R_s(z)$, $1 \le s \le M - 1$.*

The proof of Theorem 3 may be found in [2, 11]. We omit the details here. In the same way, the solution of the refinement equation (1) with symbol satisfying (14) is not always cardinal. It is not continuous and not a scaling function of a multiresolution. But when $R_s = 0$ in (14) for $1 \le s \le M - 1$, the corresponding solution is a cardinal scaling function when $N$ is sufficiently large, which we denote $\Phi_{M,N}$. Observe that

$$\binom{2N+s-1}{s} = \sum_{0\le k\le s} \binom{N-1+k}{k}\binom{N-1+s-k}{s-k}, \qquad 0 \le s \le N.$$

Therefore $\Phi_{M,2N}$ are the self-convolution of Daubechies' scaling functions $\phi_{M,N}$, but $\Phi_{M,2N+1}$ are not the self-convolution of any scaling functions. So the class of cardinal scaling functions $\Phi_{M,N}$ includes more cardinal scaling functions. Write the Sobolev exponent $s_\infty(\Phi_{M,N})$ of $\Phi_{M,N}$ as $\beta_{M,N}$. Then we have

THEOREM 4. *Let $\Phi_{M,N}$ and $\beta_{M,N}$ be defined as above. Then there exists a constant $C$ independent of $N$ such that*

$$\left|\beta_{M,N} - \frac{\ln N}{2\ln M}\right| \le C$$

*when $M$ is odd and*

$$\left|\beta_{M,N} - \frac{-N\ln\sin^2\dfrac{M\pi}{2M+2} + \ln N}{2\ln M}\right| \le C$$

*when $M$ is even.*

Until now we have considered the problem about orthonormal or cardinal scaling functions with compact support. It is natural to ask whether there exists a compactly supported scaling function which is both orthonormal and cardinal. A result in [21] tells us that such a function does not exist when $M = 2$. In Section 4, we give examples of such scaling functions for $M \ge 3$.

Let

$$H_I(z) = \frac{2 - z^M - z^{-M}}{M^2(2 - z - z^{-M})}\left(1 + \frac{M^2(\alpha+\gamma)}{2}(1 - z^{-1})(z - z^{M'+1}) + \frac{M^2(\alpha-\gamma)}{2}(1 - z)(z^{-1} - z^{-M'-1})\right),$$

where

$$\alpha = \frac{M^2-1}{12M^2M'}, \qquad \gamma = \alpha\sqrt{\frac{12M'(M'+1)}{M^2-1} - 1}$$

and $M'$ is the integral part of $(M-1)/2$. Let $\phi_I$ be the solution of (1) with the symbol $H_I(z)$. It will be shown that $\phi_I$ is a compactly supported scaling function which is both orthonormal and cardinal.

When $M = 2$, Daubechies proved in [8] that the only symmetric scaling function of a multiresolution is the Haar function. This example shows that in some sense it is possible that some properties which do not hold when $M = 2$ may hold when $M \geq 3$, or that there is more space to choose appropriate wavelet bases for $M \geq 3$ than for $M = 2$.

The paper is organized as follows. In Section 2 and 3, we give the proofs of Theorems 2 and 4, respectively. Section 4 is devoted to the construction of both orthonormal and cardinal scaling functions for $M \geq 3$.

## 2. ORTHONORMAL SCALING FUNCTIONS

In this section, we will give the proof of Theorem 2. To prove it we need some lemmas.

LEMMA 1. *Let $M \geq 3$ and*

$$f(x_1, \ldots, x_{M-1}) = \sum_{j=1}^{M-1} (1 + x_j)\ln(1 + x_j) - x_j \ln x_j - x_j \ln \sin^2 \frac{j\pi}{M},$$

*where $0 \leq x_j \leq 1$ and $\sum_{j=1}^{M-1} x_j = 1$. Then $f$ takes its maximum $2 \ln M$ at*

$$\beta_j = \frac{\sin^2 \pi/(2M)}{\sin^2 j\pi/M - \sin^2 \pi/(2M)}, \qquad 1 \leq j \leq M - 1.$$

*Proof.* By computation, we have

$$\frac{\partial^2}{\partial x_j \partial x_{j'}} f(x) = \begin{cases} -(x_j + x_j^2)^{-1} < 0, & \text{when } j = j', \\ 0, & \text{when } j \neq j'. \end{cases}$$

Therefore $f$ cannot take its maximum at the boundary of

$$\Omega = \{(x_1, \ldots, x_{M-1});\ 0 \leq x_j \leq 1,\ \sum_{j=1}^{M-1} x_j = 1\},$$

if there exists a constant $\lambda$ such that the equation

$$\left(\frac{\partial}{\partial x_j}\right) f(x_1, \ldots, x_{M-1}) = \ln \lambda, \qquad 1 \leq j \leq M - 1,$$

has a solution in the interior of the domain $\Omega$. Observe that

$$\left(\frac{\partial}{\partial x_j}\right) f(x_1, \ldots, x_{M-1}) = \ln \frac{x_j + 1}{x_j} - \ln \sin^2 \frac{j\pi}{M}.$$

Then the matter reduces to

$$\ln \frac{x_j + 1}{x_j} - \ln \sin^2 \frac{j\pi}{M} = \ln \lambda, \quad 1 \le j \le M-1, \quad \sum_{j=1}^{M-1} x_j = 1, \quad 0 < x_j < 1. \tag{15}$$

By (15), we have

$$x_j = \left( \lambda \sin^2 \frac{j\pi}{M} - 1 \right)^{-1}.$$

Hence it suffices to solve the equation about $\lambda$,

$$\sum_{j=1}^{M-1} \left( \lambda \sin^2 \frac{j\pi}{M} - 1 \right)^{-1} = 1, \qquad \lambda > 2\left( \sin \frac{\pi}{M} \right)^{-2}. \tag{16}$$

Let $u = 4\lambda^{-1}$ and

$$h(u) = \prod_{j=1}^{M-1} \left( u - 4 \sin^2 \frac{j\pi}{M} \right).$$

Then we can write (16) as

$$-uh'(u) = h(u), \qquad 0 < u < 2 \sin^2 \frac{\pi}{M}, \tag{17}$$

where $h'(u)$ denotes the derivative of $h(u)$. Let $z_0$ be a complex number such that $2 - z_0 - z_0^{-1} = u$. Then

$$h(u) = \frac{2 - z_0^M - z_0^{-M}}{2 - z_0 - z_0^{-1}}.$$

Hence the first equation in (17) can be written as

$$(-1 + z_0^{-2})^{-1}(-z_0^{M-1} + z_0^{-M-1}) = 0. \tag{18}$$

Then $z_0 \neq \pm 1$ and $z_0 = e^{\pm il\pi/M}$ for some integer $1 \le l \le M-1$ are the solutions of Eq. (18), but only $z_0 = e^{\pm i\pi/M}$ and hence $u = 4 \sin^2 \pi/2M$ is the solution of (17). Hence $f$ takes maximum at $(\beta_1, \ldots, \beta_{M-1})$, where $0 < \beta_j < 1$ is defined by

$$\beta_j = \frac{\sin^2 \pi/(2M)}{\sin^2 j\pi/M - \sin^2 \pi/(2M)}, \qquad 1 \le j \le M-1. \tag{19}$$

We now compute the maximum of $f$. By (19), we have

$$
\begin{aligned}
f(\beta_1, \ldots, \beta_{M-1}) &= \sum_{j=1}^{M-1} \ln(1+\beta_j) - \ln \sin^2 \frac{\pi}{2M} \\
&= \sum_{j=1}^{M-1} \ln\left(4 \sin^2 \frac{j\pi}{M}\right) \\
&\quad + \ln\left|\prod_{j=1}^{M-1}\left(4 \sin^2 \frac{\pi}{2M} - 4 \sin^2 \frac{j\pi}{M}\right)\right| - \ln \sin^2 \frac{\pi}{2M} \\
&= \ln M^2 + \ln \sin^2 \frac{\pi}{2M} - \ln \sin^2 \frac{\pi}{2M} = 2 \ln M.
\end{aligned}
$$

Hence Lemma 1 is proved.

LEMMA 2. *Let $a_{M,N}$ be defined by* (5). *Then there exists a constant $C$ independent of $N$ such that*

$$C^{-1} M^{2N} N^{-1/2} \le 4^{N-1} a_{M,N}(N-1) \le C M^{2N} N^{-1/2}.$$

*Proof.* By the Stirling formula $n! \sim n^n e^{-n}\sqrt{n}$, we obtain

$$\binom{N-1+s}{s} \sim s^{-1/2} \exp\Bigl((N-1)(1+y)\ln(1+y) - (N-1)y \ln y\Bigr),$$

where $0 \le y = s/(N-1) \le 1$. Hereafter two terms $A$ and $B$ are said to be equivalent to each other, denoted by $A \sim B$, if there exists a constant $C$ independent of various parameters which would take place in the formula such that

$$C^{-1} A \le B \le CA.$$

Observe that

$$4^{N-1} a_{2,N}(N-1) = \binom{2N-1}{N-1}$$

when $M = 2$. Therefore Lemma 2 holds for $M = 2$ by the Stirling formula. So it remains to prove Lemma 2 for $M \ge 3$.

By the definition of $f$ in Lemma 1 and the Stirling formula, we have

$$\prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j}\left(\sin \frac{j\pi}{M}\right)^{-2s_j} \sim e^{(N-1)f(x)} \prod_{j=1}^{M-1} s_j^{-1/2},$$

where $x = (x_1, \ldots, x_{M-1})$, $x_j = s_j/(N-1)$.

Let $\epsilon > 0$ and let $K_\epsilon$ be defined as the set of $(s_1, \ldots, s_{M-1})$ such that $\sum_{j=1}^{M-1} s_j = N$

$-1$ and $|s_j/(N-1)-\beta_j|<\epsilon$. The complement of $K_\epsilon$ in the set of all $(s_1, \ldots, s_{M-1})$ such that $\sum_{j=1}^{M-1} s_j = N-1$ is denoted $K_\epsilon^*$. Then for sufficient small $\epsilon > 0$, there exist positive constants $C_1$ and $C_2$ such that

$$C_1 \sum_{j=1}^{M-1} \left(\frac{s_j}{N-1} - \beta_j\right)^2 \le 2 \ln M - f\left(\frac{s_1}{N-1}, \ldots, \frac{s_{M-1}}{N-1}\right)$$

$$\le C_2 \sum_{j=1}^{M-1} \left(\frac{s_j}{N-1} - \beta_j\right)^2 \tag{20}$$

when $(s_1, \ldots, s_{M-1}) \in K_\epsilon$. Therefore we have

$$\sum_{(s_1,\ldots,s_{M-1})\in K_\epsilon} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j} \left(\sin \frac{j\pi}{M}\right)^{-2s_j}$$

$$\le CN^{-(M-1)/2}M^{2N} \sum_{(s_1,\ldots,s_{M-1})\in K_\epsilon} \exp\left(-C(N-1)\sum_{j=1}^{M-1}(s_j/(N-1)-\beta_j)^2\right)$$

$$\le CN^{-(M-1)/2}M^{2N} \int_{|x_j-\beta_j|\le\epsilon, 2\le j\le M-1} dx_1 \cdots dx_{M-1}$$

$$\times \exp\left(-C(N-1)\sum_{j=2}^{M-1}(x_j-\beta_j)^2 - C(N-1)\left(\sum_{j=2}^{M-1}(x_j-\beta_j)\right)^2\right)$$

$$\le CM^{2N}N^{-1/2} \int_{|x_j|<\sqrt{N}\epsilon, 2\le j\le M-1} \exp\left(-C\sum_{j=2}^{M-1}|x_j|^2 - C\left(\sum_{j=2}^{M-1}x_j\right)^2\right) dx_1 \cdots dx_{M-1}$$

$$\le \mathrm{C}M^{2N}N^{-1/2}. \tag{21}$$

For $(s_1, \ldots, s_{M-1}) \in K_\epsilon^*$, by the continuity of $f$ and the fact that $(\beta_1, \ldots, \beta_{M-1})$ is the unique point in $\Omega$ for which $f$ takes the maximum, there exists $\delta > 0$ such that

$$f\left(\frac{s_1}{N-1}, \ldots, \frac{s_{M-1}}{N-1}\right) \le 2\ln(M-\delta).$$

Hence

$$\sum_{(s_1,\ldots,s_{M-1})\in K_\epsilon^*} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j}\left(\sin\frac{j\pi}{M}\right)^{-2s_j} \le C(M-\delta)^{2N}N^{(M-1)/2}. \tag{22}$$

Combining (21) and (22), we obtain

$$\sum_{s_1+\cdots+s_{M-1}=N-1} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j}\left(\sin\frac{j\pi}{M}\right)^{-2s_j} \le CM^{2N}N^{-1/2}. \tag{23}$$

On the other hand,

$$\begin{aligned}
&\sum_{s_1+\cdots+s_{M-1}=N-1}\prod_{j=1}^{M-1}\binom{N-1+s_j}{s_j}\left(\sin\frac{j\pi}{M}\right)^{-2s_j}\\
&\quad\ge\sum_{(s_1,\cdots,s_{M-1})\in K_\epsilon}\prod_{j=1}^{M-1}\binom{N-1+s_j}{s_j}\left(\sin\frac{j\pi}{M}\right)^{-2s_j}\\
&\quad\ge CN^{-(M-1)/2}M^{2N}\sum_{(s_1,\cdots,s_{M-1})\in K_\epsilon}\exp\left(-C(N-1)\sum_{j=1}^{M-1}(s_j/(N-1)-\beta_j)^2\right)\\
&\quad\ge CM^{2N}N^{-1/2},
\end{aligned}$$

where the second inequality follows from (20) and the last one may be proved by the same computation used in (21). Hence

$$4^{N-1}a_{M,N}(N-1)\ge CM^{2N}N^{-1/2}. \tag{24}$$

This proves Lemma 2 by combining (23) and (24).

LEMMA 3. *Let $M\ge 2$ be an even integer. Then $|\sin\xi/2\sin M\xi/2|$ decreases on $[M\pi/(M+1),\pi]$.*

*Proof.* Let $g(\xi)=\tan M\xi/2+M\tan\xi/2$. Then $g'(\xi)>0$ on $[M\pi/(M+1),\pi]$ and

$$g\left(\frac{M\pi}{M+1}\right)=(M-(-1)^{M/2})\tan\frac{M\pi}{2M+2}>0.$$

By computation, we have

$$2\left(\sin\frac{\xi}{2}\sin\frac{M\xi}{2}\right)'=\cos\frac{M\xi}{2}\cos\frac{\xi}{2}\,g(\xi).$$

It is easy to see that $\cos M\xi/2\ge 0$ on $[M\pi/(M+1),\pi]$ when $M/2$ is even, and $\cos M\xi/2\le 0$ on $[M\pi/(M+1),\pi]$ when $M/2$ is odd. Then Lemma 3 follows from $\sin\xi/2\sin M\xi/2=0$ when $\xi=\pi$.

LEMMA 4. *Let $M$ be an even integer and $\xi_0=2\arcsin(\sin^2 M\pi/(2M+2))$. Then there exists a constant $C$ independent of $N$ such that*

$$C^{-1}M^{2N}N^{-1/2}\sin^{2N}\frac{\xi_0}{2}\le P_{M,N}(e^{i\xi_0})\le CM^{2N}N^{-1/2}\sin^{2N}\frac{\xi_0}{2}$$

*Proof.* By (5), we have

$$a_{M,N}(s)4^s\le\frac{1}{2}\sin^2\frac{\pi}{M}\,a_{M,N}(s+1)4^{s+1},\qquad 0\le s\le N-2.$$

Hence

$$P_{M,N}(e^{i\xi_0}) \le a_{M,N}(N-1)\sin^{2N-2}\frac{\xi_0}{2}\sum_{s=0}^{N-1}\left(\frac{\sin^2\pi/M}{2\sin^2\xi_0/2}\right)^{N-1-s} \le C_1 M^{2N} N^{-1/2}\sin^{2N}\frac{\xi_0}{2},$$

where the last inequality follows from Lemma 2 and

$$\sin^2\frac{\xi_0}{2} = \sin^4\frac{M\pi}{2M+2} > \frac{1}{2}\sin^2\frac{\pi}{M}$$

for all $M \ge 2$. On the other hand,

$$P_{M,N}(e^{i\xi_0}) \ge a_{M,N}(N-1)\sin^{2N-2}\frac{\xi_0}{2} \ge C_2 M^{2N} N^{-1/2}\sin^{2N}\frac{\xi_0}{2}$$

by Lemma 2. Hence Lemma 4 follows by letting $C = \max(C_1, C_2^{-1})$.

LEMMA 5. *Let* $H(z) = ((1-z^M)/M(1-z))^N Q(z)$ *and* $\hat{\phi}(\xi) = \prod_{n=1}^{\infty} H(e^{i\xi/M^n})$. *Define*

$$B_j = \sup_{\xi\in R}\left|\prod_{n=1}^{j} Q(e^{i\xi/M^n})\right|^2$$

*and* $b_j = \ln B_j/(2j\ln M)$ *for some positive integer j. Then*

$$|\hat{\phi}(\xi)| \le C(1+|\xi|)^{b_j-N}, \qquad \forall\xi\in R,$$

*for some constant C, and*

$$s_\infty(\phi) \ge N - b_j.$$

A similar result can be found in [5]. Lemma 5 can be proved by the same procedure used there. We omit the details here.

Now we start to prove Theorem 2 by dividing into two cases: $M$ is odd and even.

*Case 1: M is odd.* Observe that

$$0 \le P_{M,N}(e^{i\xi}) \le P_{M,N}(-1) = \sum_{s=0}^{N-1} a_{M,N}(s)4^s$$

and

$$a_{M,N}(s)4^s \le \frac{1}{2}\sin^2\frac{\pi}{M}\, a_{M,N}(s+1)4^{s+1} \le \frac{1}{2}\, a_{M,N}(s+1)4^{s+1}, \qquad 0 \le s \le N-2$$

by (5). Therefore by Lemma 2, we obtain

$$\sum_{s=0}^{N-1} a_{M,N}(s)4^s \leq 2a_{M,N}(N-1)4^{N-1} \leq CM^{2N}N^{-1/2}$$

and

$$\sum_{s=0}^{N-1} a_{M,N}(s)4^s \geq a_{M,N}(N-1)4^{N-1} \geq C^{-1}M^{2N}N^{-1/2}.$$

Hence by Lemma 5 we have

$$\alpha_{M,N} \geq \frac{\ln N}{4 \ln M} - C.$$

On the other hand, there exists a constant $C$ independent of $j$ by (5) such that

$$|\widehat{\phi_{M,N}}(M^j\pi)|^2 = (M^{-2N}P_{M,N}(-1))^j|\widehat{\phi_{M,N}}(\pi)|^2 \geq (CN^{-1/2})^j, \qquad j \geq 1.$$

Hence

$$\alpha_{M,N} \leq \frac{\ln N}{4 \ln M} + C$$

by Lemma 5 and the conclusion of Theorem 2 holds for the case $M$ is odd.

*Case 2: M is even.* By the definition of $P_{M,N}$, we have

$$P_{M,N}(e^{i\xi}) \leq \max\left(P_{M,N}(e^{i\xi_0}),\, P_{M,N}(e^{i\xi_0})\left(\frac{\sin^2\xi/2}{\sin^2\xi_0/2}\right)^{N-1}\right),$$

where $\xi_0 = 2\arcsin(\sin^2(M\pi/2M+2))$. Therefore

$$\begin{aligned}|P_{M,N}(e^{i\xi})| &\leq P_{M,N}(e^{i\xi_0})\left(\sin\frac{\xi_0}{2}\right)^{-2N+2}\sin^{2N-2}\frac{M\pi}{2M+2} \\ &\leq CM^{2N}N^{-1/2}\sin^{2N-2}\frac{M\pi}{2M+2} \qquad (25)\end{aligned}$$

by Lemma 4 when $|\xi| \leq M\pi/(M+1)$ and

$$\begin{aligned}|P_{M,N}(e^{i\xi})P_{M,N}(e^{iM\xi})| &\leq (P_{M,N}(e^{i\xi_0}))^2\left(\frac{\sin^2\xi/2}{\sin^2\xi_0/2}\right)^{N-1}\max\left(1,\left(\frac{\sin^2 M\xi/2}{\sin^2\xi_0/2}\right)^{N-1}\right) \\ &\leq (P_{M,N}(e^{i\xi_0}))^2\left(\sin\frac{\xi_0}{2}\right)^{-2N+2}\max\left(1,\left(\sin\frac{\xi_0}{2}\right)^{-2N+2}\right.\end{aligned}$$

$$\times \sup_{M\pi/(M+1)\le|\xi|\le\pi}\left(\sin\frac{\xi}{2}\sin\frac{M\xi}{2}\right)^{2N-2}\Bigg)$$

$$\le (P_{M,N}(e^{i\xi_0}))^2\left(\sin\frac{\xi_0}{2}\right)^{-2N+2} \le CM^{4N}N^{-1}\left(\sin\frac{M\pi}{2M+2}\right)^{4N-4} \tag{26}$$

when $M\pi/(M+1) \le |\xi| \le \pi$, where the third inequality follows from Lemma 3 and the definition of $\xi_0$, and the last one holds because of Lemma 4. Combining (25) and (26), we have

$$B_N = \sup_{\xi\in R}\prod_{n=0}^{N-1} P_{M,N}(e^{i\xi/M^n}) \le \left(CM^{2N}N^{-1/2}\sin^{2N}\frac{M\pi}{2M+2}\right)^N M^{2N},$$

where $B_N$ is defined as in Lemma 5. Therefore

$$\alpha_{M,N} \ge \frac{\ln N - 4N\ln\sin\dfrac{M\pi}{2M+2}}{4\ln M} - C$$

by Lemma 5. On the other hand

$$\left|\widehat{\phi_{M,N}}\left(\frac{M^{2j+1}\pi}{M+1}\right)\right|^2 = (M^{-2N}P_{M,N}(e^{iM\pi/(M+1)}))^j\left|\widehat{\phi_{M,N}}\left(\frac{M\pi}{M+1}\right)\right|^2$$

$$\ge \left(CN^{-1/2}\sin^{2N}\frac{M\pi}{2M+2}\right)^{2j}$$

for every $j \ge 1$. Hence

$$\alpha_{M,N} \le \frac{\ln N - 4N\ln\sin(M\pi/(2M+2))}{4\ln M} + C$$

by Lemma 5. This proves the conclusion for the case $M$ is even and hence completes the proof of Theorem 2.

## 3. CARDINAL SCALING FUNCTIONS

In this section we will give the proof of the asymptotic regularity of cardinal scaling functions.

*Proof of Theorem 4.* By the proof of Lemma 2, there exists a constant $C$ independent of $N$ such that

$$C^{-1}M^N N^{-1/2} \le b_{M,N}\left(\left[\frac{N-1}{2}\right]\right) \le CM^N N^{-1/2},$$

where $[x]$ denotes the integral part of $x$. Also notice that for $z = e^{i\xi}$,

$$Q_{M,N}(z)Q_{M,N}(z^{-1}) = (\sum_{0\le s\le[(N-1)/2]} b_{M,N}(s)(2 - z - z^{-1})^s)^2,$$

when $M$ is odd or $N$ is even, and

$$\begin{aligned}\left(b_{M,N}\left(\frac{N-1}{2}\right)\right)^2 &(2 - z - z^{-1})^{N-1} + \frac{2 + z + z^{-1}}{2} \\ &\times \left(\sum_{0\le s\le (N+1)/2} b_{M,N}(s)(2 - z - z^{-1})^s\right)^2 \\ &\le Q_{M,N}(z^{-1})Q_{M,N}(z) \\ &\le 4\left(\sum_{0\le s\le (N-1)/2} b_{M,N}(s)(2 - z - z^{-1})^s\right)^2\end{aligned}$$

when $M$ is even and $N$ is odd. Then we may conclude Theorem 4 by the procedure to prove Theorem 2 line by line.

## 4. ORTHONORMAL AND CARDINAL SCALING FUNCTIONS

In this section we will construct examples of compactly supported scaling functions which are both orthonormal and cardinal for all $M \ge 3$.

It is proved in [21] that there does not exist a compactly supported scaling function which is both orthonormal and cardinal when $M = 2$. So we shall assume $M \ge 3$ in this section.

Define $M' = (M - 1)/2$ when $M$ is odd and $M' = M/2 - 1$ when $M$ is even. Define $H_I(z)$ by

$$\begin{aligned} H_I(z) = \frac{2 - z^M - z^{-M}}{M^2(2 - z - z^{-1})} \Bigg( 1 &+ \frac{M^2(\alpha + \gamma)}{2}(1 - z^{-1})(z - z^{M'+1}) \\ &+ \frac{M^2(\alpha - \gamma)}{2}(1 - z)(z^{-1} - z^{-M'-1}) \Bigg), \end{aligned} \tag{27}$$

where

$$\alpha = \frac{M^2 - 1}{12M^2M'} \quad \text{and} \quad \gamma = \alpha\sqrt{\frac{12M'(M'+1)}{M^2 - 1} - 1}\,.$$

For $H_I$, it is easy to check that $H_I(1) = 1$,

$$\sum_{0\le s\le M-1} H_I(ze^{2\pi is/M}) = 1 \tag{28}$$

and

$$\sum_{s=0}^{M-1} H_I(ze^{2\pi is/M})H_I(z^{-1}e^{-2\pi is/M}) = 1 + \left(2\alpha M' - \frac{M^2-1}{6M^2}\right)(2 - z^M - z^{-M})$$
$$+ MM'\left(\frac{\alpha^2+\gamma^2}{2} - \frac{\alpha(M'+1)}{2M^2}\right)(2 - z^M - z^{-M})^2 = 1, \tag{29}$$

because

$$H_I(z)H_I(z^{-1}) = M^{-4}\left(\frac{2 - z^M - z^{-M}}{2 - z - z^{-1}}\right)^2 + \alpha M^{-2}\,\frac{(2 - z^M - z^{-M})^2}{2 - z - z^{-1}}$$
$$\times \Big(\sum_{j=1}^{M'} z^j + \sum_{j=1}^{M'} z^{-j}\Big) + (2 - z^M - z^{-M})^2$$
$$\times\left(\frac{\alpha^2+\gamma^2}{4}\sum_{j=-M'+1}^{M'-1}(M' - |j|)z^j + \frac{\alpha^2-\gamma^2}{4}\Big(\Big(\sum_{j=1}^{M'} z^j\Big)^2 + \Big(\sum_{j=1}^{M'} z^{-j}\Big)^2\Big)\right).$$

Let $\phi_I$ be the compactly supported solution of the refinement equation

$$\phi_I(x) = \sum_{k\in Z} c_k\phi_I(Mx - k), \qquad \hat{\phi}_I(0) = 1, \tag{30}$$

where $\{c_k\}$ are the coefficients of $z^k$ in the symbol $H_I$, i.e.,

$$H_I(z) = \frac{1}{M}\sum_{k\in Z} c_k z^k.$$

Hence by (28) and (29), $\phi_I$ in (30) is a compactly supported scaling function which is both cardinal and orthonormal if the following statements hold,

$$H_I(e^{i\xi}) \ne 0 \tag{31}$$

when $|\xi| \le \pi/M$ and

$$|\hat{\phi}(\xi)| \le C(1 + |\xi|)^{-1-\epsilon} \tag{32}$$

for some $\epsilon > 0$.

We first check (31). Write

$$H(z) = \frac{2 - z^M - z^{-M}}{M^2(2 - z - z^{-1})} A_0(z).$$

Observe that

$$\mathrm{Im} A_0(e^{i\xi}) = 4M^2\gamma \sin\frac{\xi}{2} \sin\frac{M'\xi}{2} \sin\frac{(M'+1)\xi}{2} \neq 0$$

when $0 < |\xi| \leq \pi/M$ and $A_0(1) = 1 \neq 0$, where $\mathrm{Im}z$ denotes the imaginary part of a complex number $z$. Therefore (31) holds.

Second, we check (32). Observe that

$$\begin{aligned} |A_0(e^{i\xi})| &\leq 1 + 2M^2(\alpha + \gamma) + 2M^2|\alpha - \gamma| \\ &\leq 1 + 4\gamma M^2 \leq 1 + \frac{2\sqrt{2}}{3}(M+1) < M \end{aligned}$$

when $M \geq 35$. By direct numerical calculations, it is verified that

$$\sup_{\xi \in R} |A_0(e^{i\xi}) A_0(e^{iM\xi})| < M^2$$

when $3 \leq M \leq 35$. In fact

$$\sup_{\xi \in R} |A_0(e^{i\xi})| < M$$

when $3 \leq M \leq 35$, except when $M = 4$. Therefore (32) holds by Lemma 5. Thus we construct a class of compactly supported scaling functions which are both orthonormal and cardinal for $M \geq 3$.

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